

INTEGRAL GROUP RING OF RUDVALIS SIMPLE GROUP

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ABSTRACT. Using the Luthar–Passi method, we investigate the classical Zassenhaus conjecture for the normalized unit group of the integral group ring of the Rudvalis sporadic simple group Ru . As a consequence, for this group we confirm Kimmerle’s conjecture on prime graphs.

1. INTRODUCTION, CONJECTURES AND MAIN RESULTS

Let $V(\mathbb{Z}G)$ be the normalized unit group of the integral group ring $\mathbb{Z}G$ of a finite group G . A long-standing conjecture of H. Zassenhaus (**ZC**) says that every torsion unit $u \in V(\mathbb{Z}G)$ is conjugate within the rational group algebra $\mathbb{Q}G$ to an element in G (see [27]).

For finite simple groups the main tool for the investigation of the Zassenhaus conjecture is the Luthar–Passi method, introduced in [22] to solve it for A_5 and then applied in [23] for the case of S_5 . Later M. Hertweck in [18] extended the Luthar–Passi method and applied it for the investigation of the Zassenhaus conjecture for $PSL(2, p^n)$. The Luthar–Passi method proved to be useful for groups containing non-trivial normal subgroups as well. For some recent results we refer to [5, 7, 16, 17, 18, 19]. Also, some related properties and some weakened variations of the Zassenhaus conjecture can be found in [1, 3, 23].

First of all, we need to introduce some notation. By $\#(G)$ we denote the set of all primes dividing the order of G . The Gruenberg–Kegel graph (or the prime graph) of G is the graph $\pi(G)$ with vertices labeled by the primes in $\#(G)$ and with an edge from p to q if there is an element of order pq in the group G . In [21] W. Kimmerle proposed the following weakened variation of the Zassenhaus conjecture:

(KC) If G is a finite group then $\pi(G) = \pi(V(\mathbb{Z}G))$.

In particular, in the same paper W. Kimmerle verified that **(KC)** holds for finite Frobenius and solvable groups. We remark that with respect to the so-called p -version of the Zassenhaus conjecture the investigation of Frobenius groups was completed by M. Hertweck and the first author in [4]. In [6, 7, 8, 9, 10, 12] **(KC)** was confirmed for the Mathieu simple groups M_{11} , M_{12} , M_{22} , M_{23} , M_{24} and the sporadic Janko simple groups J_1 , J_2 and J_3 .

Here we continue these investigations for the Rudvalis simple group Ru . Although using the Luthar–Passi method we cannot prove the rational conjugacy for torsion units of $V(\mathbb{Z}\text{Ru})$, our main result gives a lot of information on partial augmentations of these units. In particular, we confirm the Kimmerle’s conjecture for this group.

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Let $G = \text{Ru}$. It is well known (see [25]) that $|G| = 2^{14} \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13 \cdot 29$ and $\exp(G) = 2^4 \cdot 3 \cdot 5 \cdot 7 \cdot 13 \cdot 29$. Let

$$\begin{aligned} \mathcal{C} = \{ & C_1, C_{2a}, C_{2b}, C_{3a}, C_{4a}, C_{4b}, C_{4c}, C_{4d}, C_{5a}, C_{5b}, C_{6a}, C_{7a}, C_{8a}, C_{8b}, \\ & C_{8c}, C_{10a}, C_{10b}, C_{12a}, C_{12b}, C_{13a}, C_{14a}, C_{14b}, C_{14c}, C_{15a}, C_{16a}, \\ & C_{16b}, C_{20a}, C_{20b}, C_{20c}, C_{24a}, C_{24b}, C_{26a}, C_{26b}, C_{26c}, C_{29a}, C_{29b} \} \end{aligned}$$

be the collection of all conjugacy classes of Ru , where the first index denotes the order of the elements of this conjugacy class and $C_1 = \{1\}$. Suppose $u = \sum \alpha_g g \in V(\mathbb{Z}G)$ has finite order k . Denote by $\nu_{nt} = \nu_{nt}(u) = \varepsilon_{C_{nt}}(u) = \sum_{g \in C_{nt}} \alpha_g$ the partial augmentation of u with respect to C_{nt} . From the Berman–Higman Theorem (see [2] and [26], Ch.5, p.102) one knows that $\nu_1 = \alpha_1 = 0$ and

$$(1) \quad \sum_{C_{nt} \in \mathcal{C}} \nu_{nt} = 1.$$

Hence, for any character χ of G , we get that $\chi(u) = \sum \nu_{nt} \chi(h_{nt})$, where h_{nt} is a representative of the conjugacy class C_{nt} .

Our main result is the following

Theorem 1. *Let G denote the Rudvalis sporadic simple group Ru . Let u be a torsion unit of $V(\mathbb{Z}G)$ of order $|u|$ and let*

$$\begin{aligned} \mathfrak{P}(u) = & (\nu_{2a}, \nu_{2b}, \nu_{3a}, \nu_{4a}, \nu_{4b}, \nu_{4c}, \nu_{4d}, \nu_{5a}, \nu_{5b}, \nu_{6a}, \nu_{7a}, \nu_{8a}, \nu_{8b}, \\ & \nu_{8c}, \nu_{10a}, \nu_{10b}, \nu_{12a}, \nu_{12b}, \nu_{13a}, \nu_{14a}, \nu_{14b}, \nu_{14c}, \nu_{15a}, \nu_{16a}, \\ & \nu_{16b}, \nu_{20a}, \nu_{20b}, \nu_{20c}, \nu_{24a}, \nu_{24b}, \nu_{26a}, \nu_{26b}, \nu_{26c}, \nu_{29a}, \nu_{29b}) \in \mathbb{Z}^{35} \end{aligned}$$

be the tuple of partial augmentations of u . The following properties hold.

- (i) *If $|u| \notin \{28, 30, 40, 48, 52, 56, 60, 80, 104, 112, 120, 208, 240\}$, then $|u|$ coincides with the order of some element $g \in G$. Equivalently, there is no elements of orders 21, 35, 39, 58, 65, 87, 91, 145, 203 and 377 in $V(\mathbb{Z}G)$.*
- (ii) *If $|u| \in \{3, 7, 13\}$, then u is rationally conjugate to some $g \in G$.*
- (iii) *If $|u| = 2$, the tuple of the partial augmentations of u belongs to the set*

$$\{ \mathfrak{P}(u) \mid \nu_{2a} + \nu_{2b} = 1, -10 \leq \nu_{2a} \leq 11, \nu_{kx} = 0, kx \notin \{2a, 2b\} \}.$$

- (iv) *If $|u| = 5$, the tuple of the partial augmentations of u belongs to the set*

$$\{ \mathfrak{P}(u) \mid \nu_{5a} + \nu_{5b} = 1, -1 \leq \nu_{5a} \leq 6, \nu_{kx} = 0, kx \notin \{5a, 5b\} \}.$$

- (v) *If $|u| = 29$, the tuple of the partial augmentations of u belongs to the set*

$$\{ \mathfrak{P}(u) \mid \nu_{29a} + \nu_{29b} = 1, -4 \leq \nu_{29a} \leq 5, \nu_{kx} = 0, kx \notin \{29a, 29b\} \}.$$

As an immediate consequence of part (i) of the Theorem we obtain

Corollary 1. *If $G = \text{Ru}$ then $\pi(G) = \pi(V(\mathbb{Z}G))$.*

2. PRELIMINARIES

The following result is a reformulation of the Zassenhaus conjecture in terms of vanishing of partial augmentations of torsion units.

Proposition 1. *(see [22] and Theorem 2.5 in [24]) Let $u \in V(\mathbb{Z}G)$ be of order k . Then u is conjugate in $\mathbb{Q}G$ to an element $g \in G$ if and only if for each d dividing k there is precisely one conjugacy class C with partial augmentation $\varepsilon_C(u^d) \neq 0$.*

The next result now yield that several partial augmentations are zero.

Proposition 2. (see [16], Proposition 3.1; [18], Proposition 2.2) *Let G be a finite group and let u be a torsion unit in $V(\mathbb{Z}G)$. If x is an element of G whose p -part, for some prime p , has order strictly greater than the order of the p -part of u , then $\varepsilon_x(u) = 0$.*

The key restriction on partial augmentations is given by the following result that is the cornerstone of the Luthar–Passi method.

Proposition 3. (see [22, 18]) *Let either $p = 0$ or p a prime divisor of $|G|$. Suppose that $u \in V(\mathbb{Z}G)$ has finite order k and assume k and p are coprime in case $p \neq 0$. If z is a complex primitive k -th root of unity and χ is either a classical character or a p -Brauer character of G , then for every integer l the number*

$$(2) \quad \mu_l(u, \chi, p) = \frac{1}{k} \sum_{d|k} \text{Tr}_{\mathbb{Q}(z^d)/\mathbb{Q}} \{ \chi(u^d) z^{-dl} \}$$

is a non-negative integer.

Note that if $p = 0$, we will use the notation $\mu_l(u, \chi, *)$ for $\mu_l(u, \chi, 0)$.

Finally, we shall use the well-known bound for orders of torsion units.

Proposition 4. (see [13]) *The order of a torsion element $u \in V(\mathbb{Z}G)$ is a divisor of the exponent of G .*

3. PROOF OF THE THEOREM

Throughout this section we denote the group Ru by G . The character table of G , as well as the p -Brauer character tables, which will be denoted by $\mathfrak{BCI}(p)$ where $p \in \{2, 3, 5, 7, 13, 29\}$, can be found using the computational algebra system GAP [15], which derives these data from [14, 20]. For the characters and conjugacy classes we will use throughout the paper the same notation, indexation inclusive, as used in the GAP Character Table Library.

Since the group G possesses elements of orders 2, 3, 4, 5, 6, 7, 8, 10, 12, 13, 14, 15, 16, 20, 24, 26 and 29, first of all we investigate units of some these orders (except the units of orders 4, 6, 8, 10, 12, 14, 15, 16, 20, 24 and 26). After this, by Proposition 4, the order of each torsion unit divides the exponent of G , so to prove the Kimmerle’s conjecture, it remains to consider units of orders 21, 35, 39, 58, 65, 87, 91, 145, 203 and 377. We will prove that no units of all these orders do appear in $V(\mathbb{Z}G)$.

Now we consider each case separately.

- Let u be an involution. By (1) and Proposition 2 we have that $\nu_{2a} + \nu_{2b} = 1$. Put $t_1 = 3\nu_{2a} - 7\nu_{2b}$ and $t_2 = 11\nu_{2a} - 7\nu_{2b}$. Applying Proposition 3 we get the following system

$$\begin{aligned} \mu_1(u, \chi_2, *) &= \frac{1}{2}(2t_1 + 378) \geq 0; & \mu_0(u, \chi_2, *) &= \frac{1}{2}(-2t_1 + 378) \geq 0; \\ \mu_0(u, \chi_4, *) &= \frac{1}{2}(2t_2 + 406) \geq 0; & \mu_1(u, \chi_4, *) &= \frac{1}{2}(-2t_2 + 406) \geq 0. \end{aligned}$$

From these restrictions and the requirement that all $\mu_i(u, \chi_j, *)$ must be non-negative integers we get 22 pairs (ν_{2a}, ν_{2b}) listed in part (iii) of the Theorem 1.

Note that using our implementation of the Luthar–Passi method, which we intended to make available in the GAP package LAGUNA [11], we computed inequalities from Proposition 3 for every irreducible character from ordinary and Brauer character tables, and for every $0 \leq l \leq |u| - 1$, but the only inequalities that really

matter are those ones listed above. The same remark applies for all other orders of torsion units considered in the paper.

- Let u be a unit of order either 3, 7 or 13. Using Proposition 2 we obtain that all partial augmentations except one are zero. Thus by Proposition 1 part (ii) of the Theorem 1 is proved.

- Let u be a unit of order 5. By (1) and Proposition 2 we get $\nu_{5a} + \nu_{5b} = 1$. Put $t_1 = 6\nu_{5a} + \nu_{5b}$ and $t_2 = 3\nu_{5a} - 2\nu_{5b}$. By (2) we obtain the system of inequalities

$$\begin{aligned}\mu_0(u, \chi_4, *) &= \frac{1}{5}(4t_1 + 406) \geq 0; & \mu_1(u, \chi_4, *) &= \frac{1}{5}(-t_1 + 406) \geq 0; \\ \mu_0(u, \chi_2, 2) &= \frac{1}{5}(4t_2 + 28) \geq 0; & \mu_1(u, \chi_2, 2) &= \frac{1}{5}(-t_2 + 28) \geq 0.\end{aligned}$$

Again, using the condition for $\mu_i(u, \chi_j, p)$ to be non-negative integers, we obtain eight pairs (ν_{5a}, ν_{5b}) listed in part (iv) of the Theorem 1.

- Let u be a unit of order 29. By (1) and Proposition 2 we have that $\nu_{29a} + \nu_{29b} = 1$. Put $t_1 = 15\nu_{29a} - 14\nu_{29b}$. Then using (2) we obtain the system of inequalities

$$\begin{aligned}\mu_1(u, \chi_6, 2) &= \frac{1}{29}(t_1 + 8192) \geq 0; & \mu_2(u, \chi_7, 5) &= \frac{1}{29}(-t_1 + 2219) \geq 0; \\ \mu_1(u, \chi_2, 5) &= \frac{1}{29}(12\nu_{29a} - 17\nu_{29b} + 133) \geq 0; \\ \mu_2(u, \chi_2, 5) &= \frac{1}{29}(-17\nu_{29a} + 12\nu_{29b} + 133) \geq 0.\end{aligned}$$

Now applying the condition for $\mu_i(u, \chi_j, p)$ to be non-negative integers we obtain ten pairs (ν_{29a}, ν_{29b}) listed in part (v) of the Theorem 1.

Now it remains to prove part (i) of the Theorem 1.

- Let u be a unit of order 21. By (1) and Proposition 2 we obtain that $\nu_{3a} + \nu_{7a} = 1$. By (2) we obtain the system of inequalities

$$\begin{aligned}\mu_1(u, \chi_4, *) &= \frac{1}{21}(\nu_{3a} + 405) \geq 0; & \mu_0(u, \chi_2, 2) &= \frac{1}{21}(12\nu_{3a} + 30) \geq 0; \\ \mu_7(u, \chi_2, 2) &= \frac{1}{21}(-6\nu_{3a} + 27) \geq 0,\end{aligned}$$

which has no integer solutions such that all $\mu_i(u, \chi_j, p)$ are non-negative integers.

- Let u be a unit of order 35. By (1) and Proposition 2 we get $\nu_{5a} + \nu_{7a} + \nu_{7b} = 1$. Put $t_1 = \nu_{5a} + \nu_{5b}$. Since $|u^7| = 5$, for any character χ of G we need to consider eight cases defined by part (iv) of the Theorem. Using (2), in all of these cases we get the same system of inequalities

$$\mu_0(u, \chi_2, *) = \frac{1}{35}(72t_1 + 390) \geq 0; \quad \mu_0(u, \chi_4, 2) = \frac{1}{35}(-96t_1 + 1230) \geq 0,$$

which has no integer solutions such that all $\mu_i(u, \chi_j, p)$ are non-negative integers.

- Let u be a unit of order 39. By (1) and Proposition 2 we have that $\nu_{3a} + \nu_{13a} = 1$. By (2) we obtain that

$$\begin{aligned}\mu_0(u, \chi_5, *) &= \frac{1}{39}(72\nu_{13a} + 819) \geq 0; & \mu_{13}(u, \chi_5, *) &= \frac{1}{39}(-36\nu_{13a} + 819) \geq 0; \\ \mu_1(u, \chi_2, *) &= \frac{1}{39}(\nu_{13a} + 377) \geq 0; & \mu_{13}(u, \chi_2, 2) &= \frac{1}{39}(-12\nu_{3a} - 24\nu_{13a} + 51) \geq 0.\end{aligned}$$

From the first two inequalities we obtain that $\nu_{13a} \in \{0, 13\}$, and now the last two inequalities lead us to a contradiction.

- Let u be a unit of order 58. By (1) and Proposition 2 we have that

$$\nu_{2a} + \nu_{2b} + \nu_{29a} + \nu_{29b} = 1.$$

Put $t_1 = 6\nu_{2a} - 14\nu_{2b} - \nu_{29a} - \nu_{29b}$, $t_2 = 11\nu_{2a} - 7\nu_{2b}$ and $t_3 = 64\nu_{2b} + 14\nu_{29a} - 15\nu_{29b}$. Since $|u^2| = 29$ and $|u^{29}| = 2$, according to parts (iii) and (v) of the Theorem we

need to consider 220 cases, which we can group in the following way. First, let

$$\begin{aligned} \chi(u^{29}) \in \{ & \chi(2a), -5\chi(2a) + 6\chi(2b), -10\chi(2a) + 11\chi(2b), \\ & -2\chi(2a) + 3\chi(2b), -8\chi(2a) + 9\chi(2b), 6\chi(2a) - 5\chi(2b), \\ & 3\chi(2a) - 2\chi(2b), 9\chi(2a) - 8\chi(2b), 4\chi(2a) - 3\chi(2b) \}. \end{aligned}$$

Then by (2) we obtain the system of inequalities

$$\begin{aligned} \mu_0(u, \chi_2, *) &= \frac{1}{58}(-28t_1 + \alpha) \geq 0; & \mu_{29}(u, \chi_2, *) &= \frac{1}{58}(28t_1 + \beta) \geq 0; \\ \mu_1(u, \chi_2, *) &= \frac{1}{58}(-t_1 + \gamma) \geq 0, \end{aligned}$$

$$\text{where } (\alpha, \beta, \gamma) = \begin{cases} (400, 412, 383), & \text{if } \chi(u^{29}) = \chi(2a); \\ (520, 292, 263), & \text{if } \chi(u^{29}) = -5\chi(2a) + 6\chi(2b); \\ (620, 192, 163), & \text{if } \chi(u^{29}) = -10\chi(2a) + 11\chi(2b); \\ (460, 352, 323), & \text{if } \chi(u^{29}) = -2\chi(2a) + 3\chi(2b); \\ (580, 232, 203), & \text{if } \chi(u^{29}) = -8\chi(2a) + 9\chi(2b); \\ (300, 512, 483), & \text{if } \chi(u^{29}) = 6\chi(2a) - 5\chi(2b); \\ (360, 452, 423), & \text{if } \chi(u^{29}) = 3\chi(2a) - 2\chi(2b); \\ (240, 572, 543), & \text{if } \chi(u^{29}) = 9\chi(2a) - 8\chi(2b); \\ (340, 472, 443), & \text{if } \chi(u^{29}) = 4\chi(2a) - 3\chi(2b), \end{cases}$$

which has no integral solution such that all $\mu_i(u, \chi_j, p)$ are non-negative integers.

In the remaining cases we consider the following system obtained by (2):

$$\begin{aligned} \mu_0(u, \chi_2, *) &= \frac{1}{58}(-28t_1 + \alpha_1) \geq 0; & \mu_{29}(u, \chi_2, *) &= \frac{1}{58}(28t_1 + \alpha_2) \geq 0; \\ \mu_0(u, \chi_4, *) &= \frac{1}{58}(56t_2 + \alpha_3) \geq 0; & \mu_{29}(u, \chi_4, *) &= \frac{1}{58}(-56t_2 + \alpha_4) \geq 0; \\ \mu_1(u, \chi_{34}, *) &= \frac{1}{58}(-t_3 + \beta_1) \geq 0; & \mu_4(u, \chi_{34}, *) &= \frac{1}{58}(t_3 + \beta_2) \geq 0, \end{aligned}$$

where the tuple of coefficients $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ depends only of the value of $\chi(u^{29})$:

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \begin{cases} (420, 392, 392, 420), & \text{if } \chi(u^{29}) = \chi(2b); \\ (320, 492, 572, 240), & \text{if } \chi(u^{29}) = 5\chi(2a) - 4\chi(2b); \\ (600, 212, 68, 744), & \text{if } \chi(u^{29}) = -9\chi(2a) + 10\chi(2b); \\ (540, 272, 176, 636), & \text{if } \chi(u^{29}) = -6\chi(2a) + 7\chi(2b); \\ (380, 432, 464, 348), & \text{if } \chi(u^{29}) = 2\chi(2a) - \chi(2b); \\ (260, 552, 680, 132), & \text{if } \chi(u^{29}) = 8\chi(2a) - 7\chi(2b); \\ (480, 332, 284, 528), & \text{if } \chi(u^{29}) = -3\chi(2a) + 4\chi(2b); \\ (500, 312, 248, 564), & \text{if } \chi(u^{29}) = -4\chi(2a) + 5\chi(2b); \\ (200, 612, 24, 24), & \text{if } \chi(u^{29}) = 11\chi(2a) - 10\chi(2b); \\ (220, 592, 752, 60), & \text{if } \chi(u^{29}) = 10\chi(2a) - 9\chi(2b); \\ (280, 532, 644, 168), & \text{if } \chi(u^{29}) = 7\chi(2a) - 6\chi(2b); \\ (440, 372, 356, 456), & \text{if } \chi(u^{29}) = -\chi(2a) + 2\chi(2b); \\ (560, 252, 140, 672), & \text{if } \chi(u^{29}) = -7\chi(2a) + 8\chi(2b), \end{cases}$$

while the pair (β_1, β_2) depends both on $\chi(u^{29})$ and $\chi(u^2)$:

	$\chi(29a)$	$\chi(29b)$	$5\chi(29a) - 4\chi(29b)$	$-2\chi(29a) + 3\chi(29b)$
$\chi(2b)$	110641, 110513	110670, 110542	110525, 110397	110728, 110600
$5\chi(2a) - 4\chi(2b)$	110321, 110833	110350, 110862	110205, 110717	110408, 110920
$-9\chi(2a) + 10\chi(2b)$	111217, 109937	111246, 109966	111101, 109821	111304, 110024
$-6\chi(2a) + 7\chi(2b)$	111025, 110129	111054, 110158	110909, 110013	111112, 110216
$2\chi(2a) - \chi(2b)$	110513, 110641	110542, 110670	110397, 110525	110600, 110728
$8\chi(2a) - 7\chi(2b)$	110129, 111025	110158, 111054	110013, 110909	110216, 111112
$-3\chi(2a) + 4\chi(2b)$	110833, 110321	110862, 110350	110717, 110205	110920, 110408
$-4\chi(2a) + 5\chi(2b)$	110897, 110257	110926, 110286	110781, 110141	110984, 110344
$11\chi(2a) - 10\chi(2b)$	109937, 111217	109966, 111246	109821, 111101	110024, 111304
$10\chi(2a) - 9\chi(2b)$	110001, 111153	110030, 111182	109885, 111037	110088, 111240
$7\chi(2a) - 6\chi(2b)$	110193, 110961	110222, 110990	110077, 110845	110280, 111048
$-\chi(2a) + 2\chi(2b)$	110705, 110449	110734, 110478	110589, 110333	110792, 110536
$-7\chi(2a) + 8\chi(2b)$	111089, 110065	111118, 110094	110973, 109949	111176, 110152

	$2\chi(29a) - \chi(29b)$	$-3\chi(29a) + 4\chi(29b)$	$-4\chi(29a) + 5\chi(29b)$
$\chi(2b)$	110612, 110484	110757, 110629	110786, 110658
$5\chi(2a) - 4\chi(2b)$	110292, 110804	110437, 110949	110466, 110978
$-9\chi(2a) + 10\chi(2b)$	111188, 109908	111333, 110053	111362, 110082
$-6\chi(2a) + 7\chi(2b)$	110996, 110100	111141, 110245	111170, 110274
$2\chi(2a) - \chi(2b)$	110484, 110612	110629, 110757	110658, 110786
$8\chi(2a) - 7\chi(2b)$	110100, 110996	110245, 111141	110274, 111170
$-3\chi(2a) + 4\chi(2b)$	110804, 110292	110949, 110437	110978, 110466
$-4\chi(2a) + 5\chi(2b)$	110868, 110228	111013, 110373	111042, 110402
$11\chi(2a) - 10\chi(2b)$	109908, 111188	110053, 111333	110082, 111362
$10\chi(2a) - 9\chi(2b)$	109972, 111124	110117, 111269	110146, 111298
$7\chi(2a) - 6\chi(2b)$	110164, 110932	110309, 111077	110338, 111106
$-\chi(2a) + 2\chi(2b)$	110676, 110420	110821, 110565	110850, 110594
$-7\chi(2a) + 8\chi(2b)$	111060, 110036	111205, 110181	111234, 110210
	$3\chi(29a) - 2\chi(29b)$	$-\chi(29a) + 2\chi(29b)$	$4\chi(29a) - 3\chi(29b)$
$\chi(2b)$	110583, 110455	110699, 110571	110554, 110426
$5\chi(2a) - 4\chi(2b)$	110263, 110775	110379, 110891	110234, 110746
$-9\chi(2a) + 10\chi(2b)$	111159, 109879	111275, 109995	111130, 109850
$-6\chi(2a) + 7\chi(2b)$	110967, 110071	111083, 110187	110938, 110042
$2\chi(2a) - \chi(2b)$	110455, 110583	110571, 110699	110426, 110554
$8\chi(2a) - 7\chi(2b)$	110071, 110967	110187, 111083	110042, 110938
$-3\chi(2a) + 4\chi(2b)$	110775, 110263	110891, 110379	110746, 110234
$-4\chi(2a) + 5\chi(2b)$	110839, 110199	110955, 110315	110810, 110170
$11\chi(2a) - 10\chi(2b)$	109879, 111159	109995, 111275	109850, 111130
$10\chi(2a) - 9\chi(2b)$	109943, 111095	110059, 111211	109914, 111066
$7\chi(2a) - 6\chi(2b)$	110135, 110903	110251, 111019	110106, 110874
$-\chi(2a) + 2\chi(2b)$	110647, 110391	110763, 110507	110618, 110362
$-7\chi(2a) + 8\chi(2b)$	111031, 110007	111147, 110123	111002, 109978

Additionally, when $\chi(u^{29}) \in \{\chi(2b), 7\chi(2a) - 6\chi(2b), -7\chi(2a) + 8\chi(2b)\}$, we need to consider one more inequality

$$\mu_1(u, \chi_2, *) = \frac{1}{58}(-6\nu_{2a} + 14\nu_{2b} + \nu_{29a} + \nu_{29b} + \gamma) \geq 0,$$

where $\gamma = \begin{cases} 363, & \text{if } \chi(u^{29}) = \chi(2b); \\ 503, & \text{if } \chi(u^{29}) = 7\chi(2a) - 6\chi(2b); \\ 223, & \text{if } \chi(u^{29}) = -7\chi(2a) + 8\chi(2b). \end{cases}$

All systems of inequalities, constructed as described above, have no integer solutions such that all $\mu_i(u, \chi_j, p)$ are non-negative integers.

- Let u be a unit of order 65. By (1) and Proposition 2 we have that

$$\nu_{5a} + \nu_{5b} + \nu_{13a} = 1.$$

Since $|u^{13}| = 5$, we need to consider eight cases listed in part (iv) of the Theorem. Put $t_1 = 3\nu_{5a} + 3\nu_{5b} + \nu_{13a}$ and $t_2 = 6\nu_{5a} + \nu_{5b} + 3\nu_{13a}$. Then using (2) we obtain

$$\begin{aligned} \mu_0(u, \chi_2, *) &= \frac{1}{65}(48t_1 + 402) \geq 0; & \mu_{13}(u, \chi_2, *) &= \frac{1}{65}(-12t_1 + 387) \geq 0; \\ \mu_0(u, \chi_4, *) &= \frac{1}{65}(48t_2 + \alpha) \geq 0; & \mu_{13}(u, \chi_4, *) &= \frac{1}{65}(-12t_2 + \beta) \geq 0, \end{aligned}$$

$$\text{where } (\alpha, \beta) = \begin{cases} (466, 436), & \text{if } \chi(u^{13}) = \chi(5a); \\ (446, 441), & \text{if } \chi(u^{13}) = \chi(5b); \\ (546, 416), & \text{if } \chi(u^{13}) = 5\chi(5a) - 4\chi(5b); \\ (486, 431), & \text{if } \chi(u^{13}) = 2\chi(5a) - \chi(5b); \\ (566, 411), & \text{if } \chi(u^{13}) = 6\chi(5a) - 5\chi(5b); \\ (506, 426), & \text{if } \chi(u^{13}) = 3\chi(5a) - 2\chi(5b); \\ (426, 446), & \text{if } \chi(u^{13}) = -\chi(5a) + 2\chi(5b); \\ (526, 421), & \text{if } \chi(u^{13}) = 4\chi(5a) - 3\chi(5b). \end{cases}$$

In all cases we have no solutions such that all $\mu_i(u, \chi_i, p)$ are non-negative integers.

- Let u be a unit of order 87. By (1) and Proposition 2 we have that

$$\nu_{3a} + \nu_{29a} + \nu_{29b} = 1.$$

Since $|u^3| = 29$, according to part (v) of the Theorem we need to consider ten cases. Put $t_1 = \nu_{29a} + \nu_{29b}$. In all of these cases by (2) we get the system

$$\mu_0(u, \chi_2, *) = \frac{1}{87}(56t_1 + 406) \geq 0; \quad \mu_{29}(u, \chi_2, *) = \frac{1}{87}(-28t_1 + 406) \geq 0,$$

that lead us to a contradiction.

- Let u be a unit of order 91. By (1) and Proposition 2 we get $\nu_{7a} + \nu_{13a} = 1$. Now using (2) we obtain non-compatible inequalities

$$\mu_0(u, \chi_2, 2) = \frac{1}{91}(144\nu_{13a} + 52) \geq 0; \quad \mu_7(u, \chi_2, 2) = \frac{1}{91}(-12\nu_{13a} + 26) \geq 0.$$

- Let u be a unit of order 145. By (1) and Proposition 2 we have that

$$\nu_{5a} + \nu_{5b} + \nu_{29a} + \nu_{29b} = 1.$$

Put $t_1 = 3\nu_{5a} + 3\nu_{5b} + \nu_{29a} + \nu_{29b}$. Since $|u^{29}| = 5$ and $|u^5| = 29$, for any character χ of G we need to consider 80 cases defined by parts (iv) and (v) of the Theorem. Luckily, in every case by (2) we obtain the same pair of incompatible inequalities

$$\mu_0(u, \chi_2, *) = \frac{1}{145}(112t_1 + 418) \geq 0; \quad \mu_{29}(u, \chi_2, *) = \frac{1}{145}(-28t_1 + 403) \geq 0.$$

- Let u be a unit of order 203. By (1) and Proposition 2 we have that

$$\nu_{7a} + \nu_{29a} + \nu_{29b} = 1.$$

Since $|u^7| = 29$, according to part (v) of the Theorem we need to consider ten cases. Put $t_1 = \nu_{29a} + \nu_{29b}$, and then using (2) in each case we obtain a non-compatible system of inequalities

$$\begin{aligned} \mu_{29}(u, \chi_2, 2) &= \frac{1}{203}(28t_1) \geq 0; & \mu_0(u, \chi_2, 2) &= \frac{1}{203}(-168t_1) \geq 0; \\ \mu_1(u, \chi_2, *) &= \frac{1}{203}(t_1 + 377) \geq 0. \end{aligned}$$

- Let u be a unit of order 377. By (1) and Proposition 2 we have that

$$\nu_{13a} + \nu_{29a} + \nu_{29b} = 1.$$

Since $|u^{13}| = 29$, we need to consider ten cases defined by part (v) of the Theorem. In each case by (2) we obtain the following system of inequalities

$$\begin{aligned} \mu_0(u, \chi_4, *) &= \frac{1}{377}(1008\nu_{13a} + 442) \geq 0; \\ \mu_{29}(u, \chi_4, *) &= \frac{1}{377}(-84\nu_{13a} + 403) \geq 0. \end{aligned}$$

which have no solution such that all $\mu_i(u, \chi_j, *)$ are non-negative integers.

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